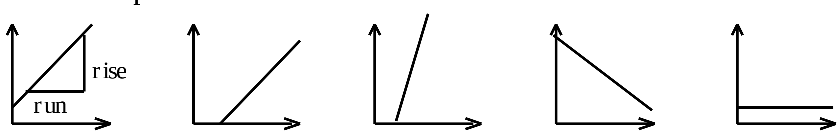
Calculus in Ten
(pages, that is)

Presented here is a quick and dirty primer to calculus. 'Quick and dirty' because we won't be proving all of the results that are discussed, but will instead rely in many places on analogy, intuition, and plausible arguments. Because of this non-rigorous approach, you will have to trust that I am not lying to you. You will see (or perhaps already have seen) the rigorous development of calculus in your math classes, including many topics that we won't touch on at all here. The purpose of these ten pages is to allow you to dive right in and start using basic calculus.

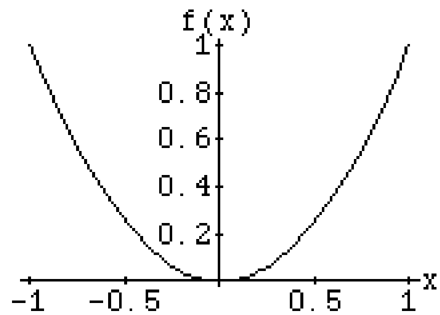
The derivative

The **slope** of a line is usually first introduced as the ratio $\frac{\text{rise}}{\text{run}}$. It describes how steep the line is. The 'run' we can better symbolize as $\Delta x \equiv x_{high} - x_{low}$, and the 'rise' as $\Delta f \equiv f(x_{high}) - f(x_{low})$. (The symbol $\equiv$ means that the two expressions are equal by definition, not because we've proven anything.) The slope is therefore $\frac{\Delta f}{\Delta x}$. The standard function of a straight line is $f(x) = mx + b$, where m is the slope. Some examples:



m=1 m=1 m=3 m=-0.8 m=0

The slope of a straight line is the same no matter where you measure it. Now, how can we define the slope of a function that is not a straight line? Let's say we have the function $f(x) = x^2$: a graph is to the right. The steepness of the function depends (unlike the case of a linear function) on where you measure it. The slope is different at x=0 than at x=0.5, which is different than at x=1 (slope is steeper) that at x=-1 (slope is negative). Let's say we want to find the slope of the curve at x=0.33. How can we determine the function's steepness there?

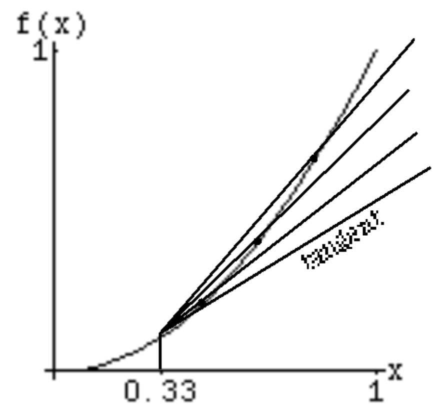


We can do that by drawing a straight line between the function at x=0.33 and some other point on the graph, at x=0.33+ε. That is shown on the right-hand graph above. The slope of that line is

$$\frac{\Delta f}{\Delta x} = \frac{f(0.33+\varepsilon) - f(0.33)}{(0.33+\varepsilon) - 0.33} = \frac{f(0.33+\varepsilon) - f(0.33)}{\varepsilon}$$

$$= \frac{(0.33+\varepsilon)^2 - (0.33)^2}{\varepsilon} = \frac{0.33^2 + 2(0.33)\varepsilon + \varepsilon^2 - 0.33^2}{\varepsilon}$$

$$= 0.66 + \varepsilon$$



Now, let's take the **limit** of this, as ε gets smaller and smaller. As ε goes to zero, Δf and Δx become infinitely small (**infinitesimal** is the adjective), and the line becomes tangent to the function, with slope equal to 0.66. We see that the slope of the function at the point x=0.66 is the slope of a straight line that is tangent to the function at that point. This limit is called the **derivative.** The process is called "taking the derivative" or, more succinctly, **differentiation.** Rather than always writing it as a limit, a new notation will be used. The derivative is written as $\frac{df}{dx}$, just like the slope of a straight line was written as $\frac{\Delta f}{\Delta x}$.

$$\left.\frac{df}{dx}\right|_x \equiv \lim_{\varepsilon\to 0}\frac{\Delta f}{\Delta x} = \lim_{\varepsilon\to 0}\frac{f(x+\varepsilon)-f(x)}{\varepsilon}$$

Let's find the derivative of $f(x)=x^2$ for <u>any</u> x (instead of only at x=0.33):

$$\frac{\Delta f}{\Delta x} = \frac{f(x+\varepsilon)-f(x)}{(x+\varepsilon)-x} = \frac{f(x+\varepsilon)-f(x)}{\varepsilon}$$

$$= \frac{(x+\varepsilon)^2-(x)^2}{\varepsilon} = \frac{x^2+2x\varepsilon+\varepsilon^2-x^2}{\varepsilon}$$

$$= 2x+\varepsilon$$

$$\frac{df}{dx} \equiv \lim_{\varepsilon\to 0}\frac{\Delta f}{\Delta x} = \lim_{\varepsilon\to 0}(2x+\varepsilon) = 2x$$

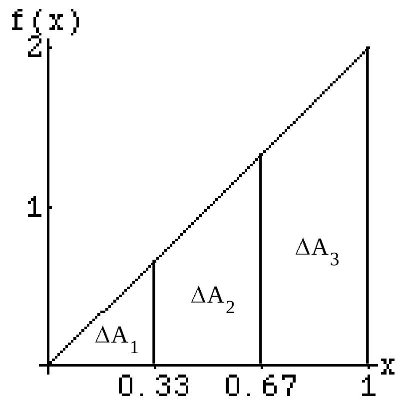
Here we can see that the slope of x^2 at x=1 is 2, at x=0.5 the slope is 1.0, and at x=0.33 the slope is 0.66 (as we found above.) In the real world, one rarely goes through that whole process to find the derivative of a function — you can learn a few rules of differentiation that allow you to easily find the derivative for many different functions. We'll come back to these techniques after we've discussed **integration**.

<u>Exercises</u> At this point, please do exercises 1 and 2.

<u>The Integral</u>

Consider the function f(x)=2x. Here's a graph of it, from x=0 to x=1. (The slope of the line is 2, even though it doesn't appear so on the graph: the scales are different on the two axes.) Let's say we want to find the area under the graph of f(x), from x=0 to x=1. We can do that by adding together the areas of the slices ΔA_1, ΔA_2, and ΔA_3. Clearly, the total area A is just the sum of the individual slices: $A=\Delta A_1+\Delta A_2+\Delta A_3$. Another way to write that is using the **summation** symbol, an upper-case Greek *sigma*.

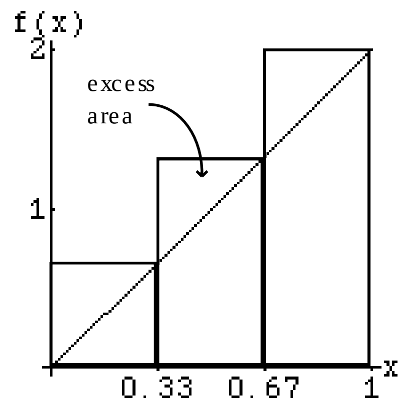
$$A = \sum_{i=1}^{3}\Delta A_i \, .$$



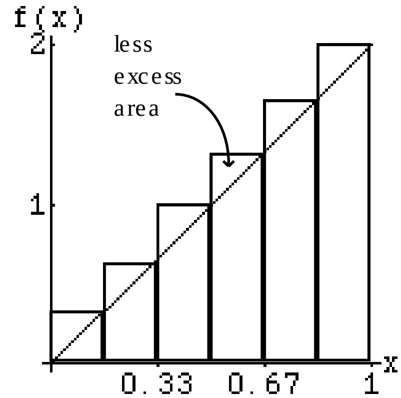
How do we know what those areas are? We could use the equation for the area of a trapezoid for each, but that is awfully complicated. Let's fudge a little, and just pretend that the area of each slice is the width of the slice, Δx, times the height, f(x), as if it were a rectangle. Now another question arises: what do we use for the height of each "rectangle", the right-hand edge or the left-hand edge? Let's not worry about it at the moment, and just use the right-hand edge. (We'll soon see that it doesn't matter.)

The area of each rectangle is $\Delta A_i = f(x_i)\Delta x$. If we're dividing the width into three pieces, then $\Delta x = \frac{1}{3}$, and x_1=0.33, x_2=0.67, and x_3=1.00. Then

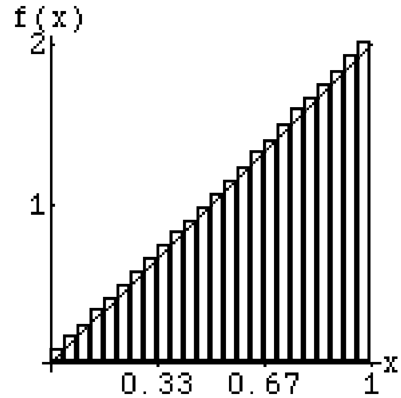
$$A \approx \sum_{i=1}^{3}\Delta A_i$$



The ≈ symbol means "approximately equal to."



It is apparent that by adding together the areas of these rectangles that we are overestimating the area under the curve: each rectangle has that small dog-ear above the function. That's the reason we used the "approximately equal to" symbol. We can alleviate this problem by, instead of using only three rectangles, using six instead. $A \approx \sum_{i=1}^{6} \Delta A_i$. Each rectangle's area is still $\Delta A_i = f(x_i)\Delta x$, but now $\Delta x = \frac{1}{6}$, and there are twice as many slices, each of which is half as wide. The excess area of the rectangles is less, and their sum is a better approximation to the actual area underneath the function.



We can do still better by using even more rectangles, say 12, or 24 (as shown to the right). Why stop there? Carrying this process to the extreme, we can add together an infinite number of rectangles. Each one of these rectangles is infinitely thin, and therefore has zero area.

Technically, the phrasing I am using here is a mathematical travesty. What we're really doing is taking a limit of the sum, as the number of rectangles increases without bound: $A = \lim_{n\to\infty} \sum_{i=1}^{n} \Delta A_i$. But my crude language does have advantages (brevity, most importantly) so I will continue to use it. If an individual rectangle becomes infinitely thin as we increase their number, there can be no difference between its height on the left-hand and right-hand edges — so it really did not matter which side we used for the height of the slice.

Recall that d's are used (instead of Δ's) to denote infinitesimal quantities. The area underneath the graph is the sum of infinitely many slices, each of which has infinitesimal area. The infinitesimal area of each 'rectangle' is dA. Rather than using Σ, we use a different symbol for the summation process —

$$A = \int dA$$

What is dA? The area of each rectangle is its height, f(x), times its width. Originally we called the width Δx, but because the number of rectangles is now infinite, the width of each one is infinitesimal — namely, dx. The area, then, must be $dA = f(x)dx$. Substituting in this expression for dA, we have

$$A = \int f(x)dx$$

We can use the equals sign now, because this is an exact determination of the area, not an approximation. The ∫ symbol was originated by Gottfried Wilhelm Leibniz (who, along with Isaac Newton, is considered the co-inventor of calculus) in 1675. It is a tall form of the letter *s* , for 'summation.'

The Connection Between Integrals and Derivatives

Recall that the (infinitesimal) area of each tiny rectangle was its height times its width:

$$dA = f(x)dx$$

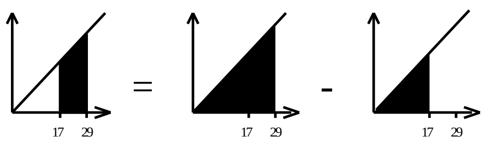
If we divide both sides by dx, we have the derivative of the area.

$$\frac{dA}{dx} = f(x)$$

Let's now return to our example, $\frac{dA}{dx} = f(x) = 2x$. Well, what function has the derivative 2x? It was the function we looked at when defining the derivative, x^2! The derivative of x^2 is 2x, and the integral of 2x is x^2. Clearly, integration and differentiation are inverse operations, like addition and subtraction, or sin and arcsin. The function x^2 is a graph of the area under the line f(x)=2x from zero to x. So the area from x=0 to x=1 is 1. It makes sense that x^2 gets bigger going to the right, because there's more area under the line 2x from zero to larger and larger x's. It also makes sense that x^2 get steeper, because as you move rightward under the line f(x)=2x, you are adding increasingly more area for every step to the right.

Definite *versus* Indefinite Integrals

In regard to integrals, I've left an important distinction vague up 'til now: that between **definite** and **indefinite** integrals. If you want to find the area under the function f(x), clearly you need to specify the left-hand and right-hand boundaries of that area. In the example I was using, f(x)=2x, the unstated assumption was that these boundaries were x=0 on the left, and x=1 on the right, because that is the domain of the function I chose to graph. Suppose instead that we wanted to find the area under the curve from x=17 to x=29. We can do that. Clearly, it would be the area from 0 to 29, minus the area from 0 to 17.



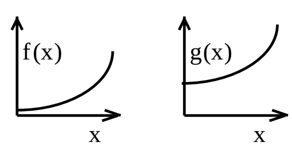
Thus, the area between x=17 and x=29 is 29^2-17^2=552. The full and proper notation that would be used is this:

$$\int_{17}^{29} 2x\,dx = x^2\Big|_{17}^{29} = 29^2 - 17^2 = 552$$

This is called a **definite integral**, because we have specified a particular domain of f(x) under which to find the area. If all you want is the function describing the area under the curve, without specifying a particular range, that is the **indefinite integral.** For f(x)=2x, the indefinite integral is just x^2.

Exercise Do exercise 3.

The Constant of Integration



The two functions at right have the same derivative. This should make sense — the slopes of the two functions are identical everywhere, the only difference being that one is higher than the other. In more explicit terms, g(x) = f(x)+C, where C is some constant number. Apparently, adding a constant to a function does nothing to its derivative. This leads us to two statements:

(a) The derivative of a constant is zero.

(b) The derivative of the sum of two functions is the sum of their derivatives.

There is never any ambiguity when taking the derivative of a function. Not so when integrating. Let's look at our example f(x)=2x. The integral of this could be x^2, or x^2+1, or x^2+17, or... you get the idea. Each of those has the same derivative, namely 2x. So the best we can do, without additional information, is to say that the indefinite integral of f(x) is x^2+C, where C is the **constant of integration.**

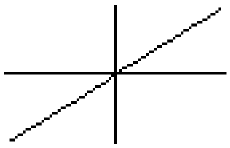
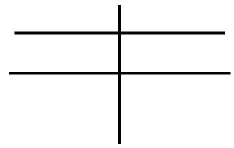
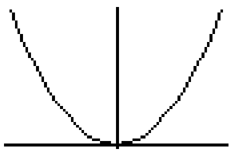
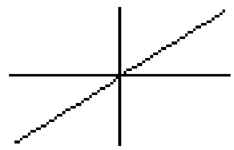
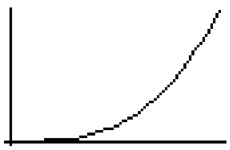
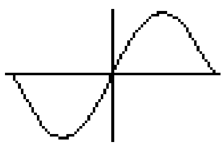
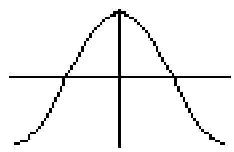
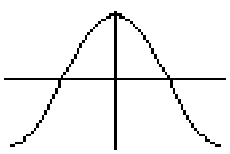
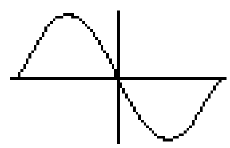
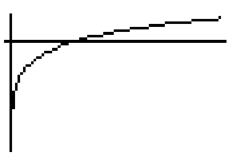
Next is a table of the most basic derivatives and integrals. They are often needed in physics.

Table of Basic Integrals and Derivatives

In the following table, 'a' and 'C' represent constants.

differentiation $\Rightarrow$

$\Leftarrow$ integration

$a \cdot x + C$			a
$x^2 + C$			$2x$
$x^n + C$			$n \cdot x^{n-1}$
$\sin x + C$			$\cos x$
$\cos x + C$			$-\sin x$
$e^x + C$			e^x
$\ln x + C$			$\frac{1}{x}$

Tools for Differentiation and Integration

Usually you aren't in need of differentiating or integrating exactly one of the functions given in the table. Instead of f(x)=sin x, it might be f(x)=3sin(5x+7)+x^2, or some such. Here are some rules to help you deal with these situations. I won't bother proving them; you'll do that in math class.

THE SUM RULE

The derivative of the sum of two functions is the sum of their derivatives. Ditto for integrals.

$$\frac{d[f(x)+g(x)]}{dx}=\frac{df}{dx}+\frac{dg}{dx} \qquad \int[f(x)+g(x)]dx=\int f(x)dx+\int g(x)dx$$

Example

Let $f(x)=x^2$ and $g(x)=\sin x$. Then $\frac{d[x^2+\sin x]}{dx}=2x+\cos x$

THE PRODUCT RULE

The derivative of the product of two functions is the sum of each times the derivative of the other.

$$\frac{d[f(x)g(x)]}{dx}=f(x)\frac{dg}{dx}+g(x)\frac{df}{dx}$$

Example

Let $f(x)=x^2$ and $g(x)=\sin x$. Then $\frac{d[x^2\sin x]}{dx}=x^2\cos x+2x\sin x$

THE CHAIN RULE

Say f is a function of g, and g is a function of x. Then

$$\frac{df(g(x))}{dx}=\frac{df}{dg}\cdot\frac{dg}{dx}$$

Example

Let $f(x)=\sin^2 x.$ Another way to write this is $f(g)=g^2,\ g(x)=\sin x.$

Then $\frac{df}{dx}=\frac{df}{dg}\frac{dg}{dx}=2g\cos x=2\sin x\cos x$.

INTEGRATION BY SUBSTITUTION

Say f is a function of g, and g is a function of x, and you are trying to integrate f with respect to x. That is, you have $\int f(g(x))dx.$ To accomplish this, first you need to get the integral into the form $\int f(g)dg$. This can often be done with a substitution, which is best illustrated by example.

Example

You are trying to integrate $\int \sin(3x)dx$. In this case, f(g)=sin g and g(x)=3x.

Now, $\frac{dg}{dx}=3$, so $dx=\frac{dg}{3}$. You can now replace dx in the integral and finish:

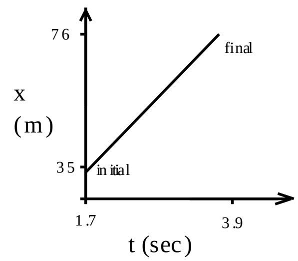
$$\int \tfrac{1}{3}\sin g\,dg=-\tfrac{1}{3}\cos g=-\tfrac{1}{3}\cos 3x.$$

Exercises Now do exercises 4, 5 and 6.

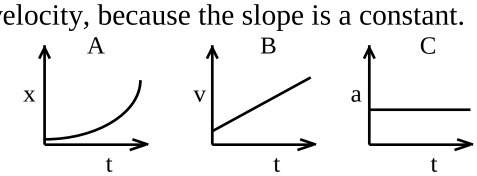
<u>Units & Physics</u>

It's time now to touch base with the real world. This is a <u>physics</u> class, after all! As physicists, we use variables to represent measurable quantities, and these have units. For instance, if both f(x) and x represent lengths along the vertical and horizontal axes, then they will have units of centimeters or meters or, if we're feeling especially barbaric, inches or feet. Let's say we're using cm. The infinitesimal interval dx will be in cm as well. When we take an integral, we multiply x by dx, yielding units of cm^2, which is a proper unit for an area.

There is no reason the two axes of a graph have to have the same units. In physics we are often interested in some property (location, temperature, *etc.*) as a function of <u>time</u>. Let's look at the position of a car as a function of time, x(t). Now x is the **dependent variable** and time, t, is the **independent variable**. (Up until now we've been using f as the dependent and x as the independent variables.) A graph of a hypothetical car's position is given to the right. Be careful — the car's position x is on the <u>vertical</u> axis. Notice also that the point where the horizontal and vertical axes cross is not (0,0): there's no law that says it has to be. What does the slope, or derivative, of this curve represent? The slope now is $\frac{\Delta x}{\Delta t}$, where $\Delta x \equiv x_{final} - x_{initial}$ is the distance the car traveled and $\Delta t \equiv t_{final} - t_{initial}$ is the amount of time it took to do so. The ratio of the two is the **rate of change** of the position. Distance per time is a **velocity**. ('Per' always means 'divided by'.) For this example, $\Delta x = 41m$ and the time interval is $\Delta t = 2.2 sec$, yielding a velocity of (with two significant digits) 19 m/sec. The car was moving with a uniform (constant) velocity, because the slope is a constant.



If the velocity is not constant, we have to take a derivative to find the velocity as a function of time, v(t). The derivative of an x-versus-t graph is written $\frac{dx}{dt}$, "the derivative of x with respect to t." So the velocity is the derivative of the position. If the position graph x(t) looks like graph A (right), then the velocity as a function of time v(t) looks like graph B. And what does the derivative of v with respect to time, $\frac{dv}{dt}$, represent. It is the <u>rate of change</u> of the velocity, also known as **acceleration**. Because v(t) as shown in graph B was a straight line, the acceleration must be a constant.



Let's pause here and consider the units. If x is measured in meters and time in seconds, then $v(t) = \frac{dx}{dt}$ has units $\frac{m}{sec}$, and acceleration has units $^{m}/_{sec}/_{sec}$ or $\frac{m}{sec^2}$.

If v(t) is the derivative of x(t), then x(t) must be the integral of v(t). Similarly with velocity and acceleration: because a(t) is the derivative of v(t), then v(t) is the integral of a(t). More simply stated:

- Acceleration is the slope of the graph v(t)
- Velocity is the slope of the graph x(t), and is also the area under the curve a(t).
- Position is the area under the graph v(t)

These relationships can be summarized in a diagram...

differentiation ⇒
$x(t) \Leftrightarrow v(t) \Leftrightarrow a(t)$
⇐ integration

Let's see what v(t) and x(t) are for the case when a(t) is a constant:

$$a(t) = a = constant$$

Memorize that; you will need it. To find v(t), we need to integrate again...

$$v(t) = \int a \cdot dt = at + C$$

We don't know offhand what the constant of integration is, but let's think about it. At t=0, the term $a \cdot t$ is also zero. Therefore, C must represent whatever velocity the object had at the start. Instead of C, let's call that initial velocity v_o:

$$v(t) = v_o + at$$

To find x(t), we need to integrate one more time...

$$x(t) = \int v(t)dt = \int (at + v_0)dt = \tfrac{1}{2}at^2 + v_0 t + C$$

Again, consider the situation at t=0. At that time, x(0)=C: in order words, this constant of integration is the initial position of the object. Call it x_o. We are left with the **equation of motion** for an object moving under uniform acceleration, in one dimension. Memorize it, too.

$$x = x_o + v_o t + \tfrac{1}{2}at^2$$

__Exercise__ Go do exercise 7 now.

__One More Example from Physics__

In physics, **work** is defined as a <u>force times a distance</u>. If the force is constant, you can just multiply the force by the displacement: $W = F \cdot d$. If the force is changing, however, that won't do. Say you have a spring which you are pulling to the right. As you pull further to the right, it gets harder and harder to stretch further, i.e. F is increasing. How to find the total work done in stretching the spring from $x=x_o$ to $x=x_f$? Well, pull the spring a tiny distance Δx and multiply by the force F at the start of that Δx. This gives you a small portion ΔW of the total work W. The reason for specifying a small Δx is that F won't change much over a small displacement. Now, pull the spring a little further, multiply F (a little bigger now) by Δx, and keep going. Adding together all those ΔW's gives the total work W. I hope this process reminds you of integration, cause that's what it is. Take the limit as Δx goes to dx, and we have

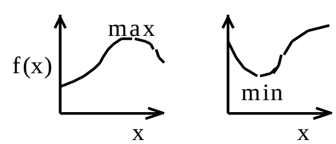
$$W = \int_{x_o}^{x_f} F(x)dx \qquad \text{units:}[W] = [F]\cdot[x] = N \cdot m \equiv J$$

That is, work is the area under the graph F(x). **Hooke's Law** says that, for a spring, F=kx, where k is a number (the **spring constant**) describing how stiff that particular spring is. Substituting that in for F(x) in the integral yields...

$$W = \int_{x_o}^{x_f} kx dx = \tfrac{1}{2}kx^2\Big|_{x_o}^{x_f} = \tfrac{1}{2}k(x_f^2 - x_o^2)$$

__Finding Maxes and Mins__

The derivative is a handy tool to find where a function has a peak value (a **maximum**) or a low value (a **minimum**). Wherever there is a max or a min to a function, the slope at that point must necessarily be zero. So to find at what values of x the maxes and mins are located, all you need do is take the derivative of the function, <u>set it equal to zero</u>, and solve for x.



<u>Example</u>

Say you want to find when a projectile is at the highest point of its trajectory. The vertical position is given by $x = x_o + v_o t + \tfrac{1}{2}at^2$. The derivative is the velocity $v = v_o + at$. It makes sense that the height, x, is maximum when the velocity is zero, because that's when the ball is momentarily between going up and coming down. Setting the derivative equal to zero yields:

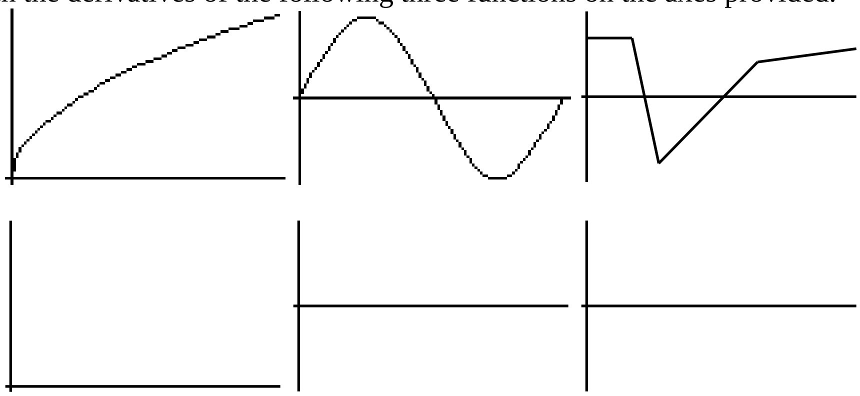
$$0 = v_o + at$$

$$t = \frac{v_o}{a}$$

For free-fall motion, a=g=-9.8 m/sec^2.

Exercises

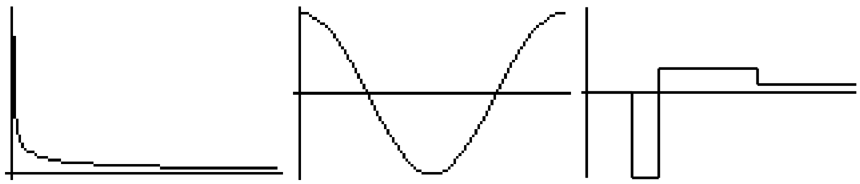
1. Using the same method I used on page 2, find the derivative of $f(x)=x^3$.
2. Sketch the derivatives of the following three functions on the axes provided.



3. Find the area under the curve $f(x)=2x$ from $x=0.70$ to 2.1.
4. Find the derivatives of (a) $1/x$ (b) $\cos(x^2)$ (c) G/x^2 (G constant)
   (d) x^4+3x^3 (e) $e^{-7x}\sin x$ (f) $\cos^2 x$
5. Find the indefinite integrals of (a), (c), and (d), above.
6. Find the definite integrals of (a), (c) and (d), from $x=1.00$ to $x=11.0$.
7. An arrow is shot upwards with an initial velocity of 32 m/sec. Ignore air resistance.
   (a) When will the arrow be at a height of 25 meters?
   (b) How fast is the arrow moving when at 25 meters?
   (c) When is the arrow at its highest point?
8. Find where the function $3x^2-x^3$ has any maxima or minima.

Solutions to Exercises

1. $\frac{df}{dx} = 3x^2$
2.



3. Area is 3.9.
4. (a) $-x^{-2}$ (b) $-2x\sin(x^2)$ (c) $-2Gx^{-3}$
   (d) $4x^3 + 9x^2$ (e) $e^{-7x}\cos x - 7e^{-7x}\sin x$ (f) $-2\cos x \cdot \sin x$
5. (a) $\ln x + C$ (c) $-\frac{G}{x} + C$ (d) $\frac{1}{5}x^5 + \frac{3}{4}x^4 + C$
6. (a) 2.40 (c) 0.909G (d) $4.32\text{x}10^4$
7. (a) At t=0.91 sec and t=5.6 sec
   (b) 23 m/sec
   (c) At t=3.3 sec
8. At x=0 and x=2.