PHY 2030 – General Relativity/Doppler Effect

1. The Schwarzschild radius (R_s) of an object is defined as the radius of a sphere such that, if all the mass (M) of the object were contained within that sphere, then the escape speed would be equal to the speed of light. If an object is smaller than its Schwarzschild radius, then it is a black hole. Using simple physics, **derive the equation** for the Schwarzschild radius,

$$R_S = \frac{GM}{c^2} \tag{1}$$

(Hint: set up $F = ma$ for an object of mass m in a circular orbit around an object of mass M.)

2. Calculate the Schwarzschild radius of the Earth (in SI units).

3. Calculate the Schwarzschild radius of the Sun (in SI units).

4. Calculate the Schwarzschild radius of the supermassive black hole at the center of the Milky Way (in SI units).

5. As light 'climbs out' of a gravitational well, its energy decreases, and its wavelength increases. This gravitational redshift in general relativity can be calculated as.

$$\frac{\Delta\lambda}{\lambda} = \frac{1}{\sqrt{1 - \frac{R_s}{r}}} - 1 \tag{2}$$

The sun's light output peaks at a wavelength of 500 nm. What wavelength do we actually observe on Earth for the sun, due to gravitational redshift?

6. You are $2R_s$ away from a black hole. If you shine a red laser away from the black hole towards Earth, what color will the laser appear to those on Earth who receive the signal?

7. The deeper an object is within a gravitational well, the more time 'slows down' compared to an object that is farther outside of the gravitational well. For an object that is a distance r from a gravitating body, the time dilation equation in general relativity is given by:

$$\tau = t\sqrt{1 - \frac{R_s}{r}} \tag{3}$$

where τ is proper time between two events as measured by a slow–ticking clock deep in a gravitational well, and t is the coordinate time measured between these events as an observer at an arbitrarily large distance away (in a flat part of spacetime). Note: This equation describes spacetime in the vicinity of a non-rotating massive spherically symmetric object.

By what factor (t/τ) does time run slower for us on the surface of the Earth, compared to someone very far away from it (in flat spacetime)?

8. Hans and Franz (identical twins) spend their entire lifetimes working inside the Empire State Building, which is 381 m tall. If Franz spends his entire life on the ground floor, and Hans spends his entire life on the top floor, they will age at different rates due to general relativity. Who will be younger at the end of his lifetime, and approximately how much younger will he be?

9. An astronaut decides to orbit a black hole with 1,000,000 times the mass of the sun at a distance of $2R_s$.

   (a) Consider only the time dilation effects of general relativity due to the astronaut's depth in the gravitational well. Every hour in the astronaut's frame is equal to how much time back home on Earth?

   (b) Consider only the time dilation effects of special relativity due to the astronaut's tremendous speed. Every hour in the astronaut's frame is equal to how much time back home on Earth?

(c) Consider BOTH effects at the same time (since both are in play!). Every hour in the astronaut's frame is equal to how much time back home on Earth?

10. (This question relates to the Doppler Effect.) You observe a hydrogen absorption line in a star that has a rest wavelength of 656 nm. However, you actually measure the wavelength to be 600 nm in the spectrum. How fast is the star moving, and is it moving towards or away from you?

11. (This question relates to the Doppler Effect.) How fast would you have to drive to see a red light as green?