PHY 2030 – Rotation Curves & Dark Matter

1. Density Profile of Dark Matter — Last class, we investigated the expected and observed rotation curves for our Milky Way galaxy. As a reminder, the density distribution of regular matter in the galaxy (stars, dust, gas, etc.) can be approximated as $\rho(r) = \rho_0 e^{-kr}$, where k and ρ_0 are constants. You should have found that, given this density distribution, the orbital velocities *should* decrease as r increases. However, observations show that the orbital velocities remain *constant* as r increases (i.e., most objects orbit the center of the Milky Way with the same speed.) This implies the presence of missing mass, which we refer to as dark matter. **Determine the density profile** $\rho(r)$ **for dark matter** required to keep the rotation curve *flat* (by following the instructions below).

   (a) Write down the triple integral (in spherical coordinates) that defines the mass $M(r)$ contained within a radius r of a galaxy with density distribution $\rho(r)$. Go ahead and simply the θ and ϕ parts, leaving on the integral over dr.

   (b) Take the derivative of both sides of the equation above with respect to r.

   (c) Solve the above equation for $\rho(r)$.

   (pause) Remind yourself of the 'bigger picture' before moving on. We know that the actual rotation curve of the galaxy is *flat*. There's more mass in the galaxy than meets the eye (dark matter). We're trying to determine the density profile $\rho(r)$ of dark matter required in order to make the rotation curve flat. (Is dark matter concentrated more in the center? The outskirts? Is is distributed evenly?)

(d) Now that you have an equation for $\rho(r)$, we must get it in terms of r only. Specifically, let's find an equation for $\frac{dM(r)}{dr}$ so we can replace it in your above equation for $\rho(r)$. Start with $F = ma$ for stars in circular orbits around a total gravitating mass $M(r)$ at a distance r. Remember, the observed rotation curve is flat for the Milky Way, so $v(r) = constant = V$. **Solve for the total mass** contained within an orbital radius r. You should get $M(r) = \frac{V^2 r}{G}$.

(e) Take the derivative of the above equation with respect to r.

(f) Re-write your equation for $\rho(r)$ from part (c), but replace $\frac{dM(r)}{dr}$ with your solution from part (d). This equation describes the mass distribution (density as a function of distance) of dark matter in our Galaxy.

(g) Sketch the mass distribution of *normal* matter in our galaxy as a function of distance r (ρ vs r).

(h) Sketch the mass distribution of *dark matter* in our galaxy as a function of distance r (ρ vs r).

2. We have seen that, in order to keep the rotation curve of the Milky Way flat (constant velocity), there must be another form of matter we refer to as dark matter. You now have the mass distribution $\rho(r)$ for dark matter. Let's test it. Beyond the visible edge of the Milky Way, there's no more regular matter, only dark matter, and the mass distribution is given by:

$$\rho(r) = \rho_{\mathrm{d}} \frac{1}{r^2} \tag{1}$$

Verify that the above gives you a flat rotation curve, starting with F=ma.

3. THE EXPANSION OF THE UNIVERSE — Edwin Hubble found that essentially all galaxies appear to be moving away from us on Earth with velocities that are linearly dependent upon their distances:

$$v = H_0 r \tag{2}$$

where H_0 is Hubble' constant ($\sim$ 70 km/s/Mpc) and r the distance. The farther away a galaxy is, the faster is moves away from us. This observation can be explained if the universe is expanding (the fabric of spacetime is 'stretching' between all objects). At all times, however, gravity tries to pull everything back together (wants to contract the universe). This raises the question: will the universe continue to expand forever, or contract?

(a) The Universe is mostly empty space, which might suggest that a Newtonian description of gravity (which is valid in the weak gravity limit) is adequate for describing the large-scale structure of the Universe. Let's pretend the universe has a uniform distribution of mass with constant density, ρ. Consider a galaxy with mass m at a distance r from the Earth (that is moving away from Earth). Show that the potential energy V of this galaxy is

$$V = -\frac{4}{3}G\pi r^2 \rho m \tag{3}$$

(b) Show that the kinetic energy K of this galaxy is

$$K = \frac{1}{2}mH_0^2 r^2 \tag{4}$$

(c) Show that the total energy of this galaxy is

$$E = \frac{1}{2} m r^2 (H_0^2 - \frac{8}{3}\pi G \rho) \tag{5}$$

(d) If the density ρ is too high, the expansion will stop, and the universe will begin to contract (the galaxy will start moving back towards earth). If the density ρ is too low, the expansion will continue forever. **Calculate the critical density ρ_{crit} that will *just* halt the expansion**

(e) The above critical density turns out to be six hydrogen atoms per cubic meter of space! We can introduce a **total density parameter** Ω defined by

$$\Omega = \frac{\rho}{\rho_{\text{crit}}} \tag{6}$$

where ρ is the total density of the universe (including all forms of mass–energy). There are three possibilities for Ω and thus the fate of our universe:

- If $\Omega > 1$ the Universe is said to be **closed** and the expansion will stop in a finite amount of time.
- If $\Omega < 1$ the Universe is said to be **open** and the expansion will never halt.
- If $\Omega = 1$ the Universe is said to be **flat** (or Euclidean) and the expansion will halt, but only asymptotically as t approaches infinity.