# PHY 2030 – Rotation Curves & Dark Matter

TRIPLE INTEGRALS — Let's review volume integrals in cylindrical and spherical coordinates.

Triple integral in cylindrical coordinates:

$$V(r, \phi, z) = \int_0^Z \int_0^{2\pi} \int_0^R r \,\mathrm{d}r \,\mathrm{d}\phi \,\mathrm{d}z \tag{1}$$

Triple integral in spherical coordinates:

$$V(r, \theta, \phi) = \int_0^{2\pi} \int_0^{\pi} \int_0^R r^2 sin\theta \,\mathrm{d}r \,\mathrm{d}\theta \,\mathrm{d}\phi \tag{2}$$

1. Use equation (1) to derive the volume of a cylinder with length $L$ and radius $R$.

2. Use equation (2) to derive the volume of a sphere with radius $R$.

3. Derive the volume of a spherical shell with inner radius $r$ and outer radius $R$.

4. Derive the mass of a solid cylinder with constant density of $\rho = \rho_0$ (assume it has height $z$ and radius $r$).

5. Derive the mass of a solid cylinder whose density increase increases away from the center as $\rho = \rho_0 r$ (assume it has height $z$ and radius $R$; $\rho_0$ is a constant). Where is most of the mass in this system concentrated?

Theoretical Rotation Curves — The **rotation curve** for a system of gravitating objects (like a galaxy) is a plot of the speeds of stars or other orbiting objects versus their radial distance from the center. We will now investigate common 'shapes' of rotation curves.

6. Consider the planets orbiting the sun in our solar system. In this system, you can assume all of the mass is located in the sun (a point mass) at $r = 0$. Derive an equation for the speed of a planet $v(r)$, where $r$ is the orbital radius and $M_{\text{sun}}$ is the mass of the sun. Assume circular orbits.

   Informed by your equation, sketch the rotation curve for the solar system.

   Describe in words the mass distribution and the shape of the rotation curve.

7. Consider a nearly–flat, symmetrical galaxy with constant density $\rho = \rho_0$, height $h$ and radius $R$. Derive an equation for the speed of orbiting stars as a function of distance, $v(r)$. Assume circular orbits.

Informed by your equation, sketch the rotation curve for $0 \leq r \leq 2R$ (from the center of the galaxy out to twice its radius). Think carefully about what should happen to the orbital velocities at $r \geq R$.

Describe in words the mass distribution and the shape of the rotation curve.

8. Our Milky Way galaxy is a relativity flat, nearly symmetrical galaxy with most of its mass concentrated near the center. Its density distribution can be approximated as $\rho(r) = \rho_0 e^{-kr}$, where $k$ and $\rho_0$ are constants. (This distribution comes from the distribution of visible stars and gas clouds.) Derive an equation for the speed of orbiting stars as a function of distance, $v(r)$. Assume circular orbits.

Informed by your equation, sketch the rotation curve (from the center of the galaxy out to twice its radius).

Describe in words the mass distribution and the shape of the rotation curve.

The Milky Way's Rotation Curve — The disk of our Galaxy is full of cold hydrogen gas that allows us to measure velocities and determine the rotation curve empirically.

9. Consider electrons in the ground state of a hydrogen atom. The spins of the electrons can either be aligned or anti–aligned with those of their protons. These two states, which should both have an energy of 13.6 eV, actually differ in energy by $5.871 \times 10^{-6}$ eV. Cold hydrogen in the Milky Way frequently emits light when it transitions from the higher energy state (spins parallel) to the lower energy state (spins anti–parallel). Calculate the wavelength of light emitted by cold hydrogen.

10. The table below presents orbital velocity measurements for various cold hydrogen gas clouds orbiting the Milky Way at various distances. Calculate the velocity of each cloud using the Doppler equation.

| Distance (kly) | $\lambda_{\text{measured}}$ (cm) | Velocity (km/s) |
|---|---|---|
| 2 | 21.1235 | |
| 3 | 21.127 | |
| 4 | 21.134 | |
| 20 | 21.135 | |
| 30 | 21.136 | |
| 50 | 21.135 | |

11. Using the data above, make a rough sketch of velocity versus distance for the Milky Way. Draw a smooth line through the data points to represent the full rotation curve. Label this line "observed" as it represents the actual rotation curve of the Milky Way.

12. Add to the above plot a sketch of the *expected* rotation curve of the Milky Way based off of the distribution of known mass (see answer to part 8); label it "expected."

13. Compare the observed rotation curve to the expected one. What do the differences mean concerning the true mass distribution in the Milky Way? Add to the above plot a line representing the rotation curve of this missing mass and label it 11missing."

14. How might you describe this missing mass? What qualities must it have?