Astronomy 121 Math Review

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Although this course will focus on conceptual understanding rather than problem solving, we do use math to do a few calculations. As a result, it will be helpful to review the following mathematical principles.

The following text is from *Astronomy 101 Math Review* by Robert Knop, Vanderbilt University and is used with his permission.

1 Exponents

An exponent is when you raise a number to a given power. You are probably most familiar with raising a number to the power of 2, which is exactly the same as squaring a number. When you raise a number to a given power, you multiply that number by itself that many times. In other words:

$$a^1 = a$$

$$a^2 = a \times a$$

$$a^3 = a \times a \times a$$

$$a^4 = a \times a \times a \times a$$

and so forth. a^3 is *not* equal to $3a$ (note that $3a$ is shorthand for $3 \times a$). As an example, suppose $a = 2$. Then $2^2 = 4$, $2^3 = 2 \times 2 \times 2 = 8$, $2^4 = 2 \times 2 \times 2 \times 2 = 16$, etc.

Sometimes you will see a number raised to a negative power. This means a fraction with 1 in the numerator, and the number raised to the positive power in the denominator, i.e.:

$$a^{-n} = \frac{1}{a^n}$$

If we continue the concrete example of $a = 2$, $2^{-1} = \frac{1}{2} = 0.5$, $2^{-2} = \frac{1}{4} = 0.25$, $2^{-3} = \frac{1}{8} = 0.125$, etc. It is often convenient, instead of writing fractions, to just write whatever *would* appear in the denominator as something with a negative power. For example, meters per second is a unit of speed— how fast you're moving. If "m" indicates a meter and "s" indicates a second, both of the following are valid ways of writing one meter per second:

$$1\ \frac{\mathrm{m}}{\mathrm{s}} = 1\ \mathrm{m\ s^{-1}}$$

Any number raised to the zeroth power is equal to 1: $a^0 = 1$ for any value of a.

If you multiply two different powers of the same number together, you sum the powers, i.e.:

$$a^n \times a^m = a^{n+m}$$

So, for example,

$$2^2 \times 2^3 = (2 \times 2) \times (2 \times 2 \times 2) = 2^{2+3} = 2^5$$

The same goes if one of the two numbers is negative:

$$2^2 \times 2^{-3} = 2^{2-3} = 2^{-1} = \frac{1}{2}$$

If you divide two different powers of the same number, you subtract the powers, i.e.:

$$\frac{a^n}{a^m} = a^{n-m}$$

For example:

$$\frac{2^4}{2^2} = \frac{2 \times 2 \times 2 \times 2}{2 \times 2} = \frac{16}{4} = 4 = 2^{4-2} = 2^2$$

The same example, only suppose we don't know that $a = 2$:

$$\frac{a^4}{a^2} = a^{4-2} = a^2$$

A fractional exponent is the same as a root. The one you will see most often is a square root: $a^{\frac{1}{2}} = \sqrt{a}$ Remember that if $b = \sqrt{a}$, it means that $b \times b = a$. Using the rules for multiplying numbers raised to a given power, you can see that the definition of the 1/2 power as a square root makes sense:

$$a^{\frac{1}{2}} \times a^{\frac{1}{2}} = a^{\frac{1}{2}+\frac{1}{2}} = a^1 = a$$

You may occasionally see something raised to the $\frac{1}{4}$ power. How do you do this with your calculator? One quick way is to take a square root twice, i.e.

$$a^{\frac{1}{4}} = \sqrt{\sqrt{a}}$$

Finally, if you have a number raised to a power, all raised to another power, you multiply the two powers:

$$(a^m)^n = a^{m \times n}$$

For example,

$$\left(2^2\right)^3 = (2 \times 2)^3 = (2 \times 2) \times (2 \times 2) \times (2 \times 2) = 2^6 = 2^{2 \times 3}$$

You will most often use this when you are trying to get rid of powers, taking one side to an appropriate fractional power, i.e.:

$$(a^n)^{\frac{1}{n}} = a^{n \times \frac{1}{n}} = a^1 = a$$

For example:

$$\left(a^{\frac{2}{3}}\right)^{\frac{3}{2}} = a$$

2 Scientific (Powers-of-Ten) Notation

In astronomy (and other areas of science), we often deal with numbers that are so big that we write them in scientific notation. Scientific notation is always a number times 10 to some power. If 10 is raised to a positive number, it tells you how many zeros there are after the one, for example:

$$10^6 = 1,000,000$$

Therefore, we would write the number 3 million as:

$$3 \times 10^6$$

Note, however, that you don't necessarily write that many zeros at the end of a number! For example, if you see the number:

$$3.5 \times 10^4$$

and you wanted to write it out longhand, you would write:

$$3.5 \times 10^4 = 35,000$$

Notice that there aren't four zeros! It is more accurate to say that **the exponent on the 10 tells you how many places to move the decimal point**. We started with 3.5, and moved the decimal point four places to the right, ending up with 35,000.

Similarly, if the exponent on the 10 is negative, it tells you how many places to move the decimal point to the left. Some examples:

$$5 \times 10^{-1} = 0.5$$

$$1.24 \times 10^{-2} = 0.0124$$

$$35 \times 10^{-1} = 3.5$$

$$0.6 \times 10^{-2} = 0.006$$

When multiplying or dividing two numbers in scientific notation, it is easiest to first multiply or divide the number before the 10, and then multiply and divide the 10's together using the rules for exponents described above. A couple of examples:

$$(5 \times 10^{-3})(3 \times 10^4) = (5 \times 3)(10^{-3} \times 10^4) = 15 \times 10^{-3+4} = 15 \times 10^1 = 150.$$

$$\frac{8.0 \times 10^{-3}}{3.0 \times 10^{-5}} = \left(\frac{8.0}{3.0}\right) \times \left(\frac{10^{-3}}{10^{-5}}\right) = 2.7 \times (10^{-3\,-\,-5}) = 2.7 \times (10^{-3+5}) = 2.7 \times 10^2$$

3 Dimensionality and Units

If you were to ask me how tall I was, and I were to answer just "one point eight", that would be a meaningless answer. 1.8 whats? Inches? Feet? Light-years? The same is true for problems you will answer in this class. If you say that the mass of a star is "2", that answer is meaningless. You would have to put units on the number: "2 solar masses", or (as it is usually written) "$2M_\odot$". The general rule is: **Always put units on numbers that need them!**

Sometimes you will come across a number which is just a number, and doesn't need units. "How many times brighter is star A than star B?" You could just answer 3, if star A is three times as bright as star B. That's just a number; there are no units on it. Frequently, though, you will be saying how fast, or how bright, or how massive, or how old, or how big something is— and in any circumstance like that, your number *must* have units on it to be meaningful.

What's more, make sure you get the right units. The best way to do this is to carry units through your problem and make sure you know what units you're working in. Do not mix units! Suppose I tell you that three objects are in a line; the distance between object A and object B is 1 foot, and the distance between object B and object C is 1 meter. What is the distance between objects A and C? If you just add the distances and come out with "2" anything, you've done it wrong! Convert one of the units to be consistent with the other unit (see "The Unit Factor Method" below) before adding them. For a graphic illustration of just why this is important, see the following article:

`http://www.cnn.com/TECH/space/9909/30/mars.metric/index.html`

If I tell you a car has gone 60 miles in two hours, what is it's speed? Speed is distance divided by time:

$$v = \frac{60 \text{ mi}}{2 \text{ h}} = 30 \ \frac{\text{mi}}{\text{h}}$$

Not just "30", but "30 miles per hour".

4 The Unit Factor Method

Sometimes you will need to convert one unit to another unit. The trick for doing this: **multiply by one as many times as necessary**. You can always multiply a number by 1 without chaning that number. The secret is writing the number 1 in a particularly clever way. Here are some ways you can write the number 1:

$$1 = \left(\frac{60 \text{ min}}{1 \text{ hr}}\right)$$

$$1 = \left(\frac{2.54 \text{ cm}}{1 \text{ in}}\right)$$

$$1 = \left(\frac{1\mathrm{M}_\odot}{2 \times 10^{30} \text{ kg}}\right)$$

If you have an expression in one set of units and you need them in another set of units, you just multiply by one as many times as necessary. Cancel out units that appear anywhere on *both* the top and bottom in your huge product, and you will be left with a number and another set of units. A simple example: convert the length 2.500 yards into centimeters:

$$2.500 \text{ yd} = (2.5 \text{ yd})\left(\frac{36 \text{ in}}{1 \text{ yd}}\right)\left(\frac{2.54 \text{ cm}}{1 \text{ in}}\right) = (2.5 \times 36 \times 2.54) \text{ cm} = 228.6 \text{ cm}$$

Notice that yards (yd) appear in the numerator and the denominator, and so get cancelled out, as does inches. We're left with just cm. All we did was mutiply the value 2.5 yd by 1, so we didn't change it at all; 228.6 cm is another way of saying 2.500 yd.

Another example: suppose I tell you that the surface area of the Sun is 2.4×10^{19} square meters. How many square miles is that?

$$\left(2.4 \times 10^{19} \text{ m}^2\right)\left(\frac{100 \text{ cm}}{1 \text{ m}}\right)^2\left(\frac{1 \text{ in}}{2.54 \text{ cm}}\right)^2\left(\frac{1 \text{ ft}}{12 \text{ in}}\right)^2\left(\frac{1 \text{ mi}}{5280 \text{ ft}}\right)^2$$

Two things to notice about this. First, notice how all the unit factors are *squared.* That's because we started with meters squared at the beginning, which is meters times meters. If we're going to get rid of both of them, we have to divide by meters twice. The same then goes for all of the other units. Next, notice that everything except for the left-over miles squared cancel out. We're left with a bunch of numbers we can punch into our calculator (remembering to square things) to get:

$$\frac{(2.4 \times 10^{19})(100^2)}{(2.54^2)(12^2)(5280^2)} \text{ mi}^2 = 9.3 \times 10^{12} \text{ mi}^2$$

One more example. Sometimes you have more than one unit to convert. If I tell you that a car moves 60 miles per hour, how many meters per second is it going? (Notice here that instead of arduously multiplying out the convrersion between meters and miles as I did in the previous example, I've looked up that there are about 1609 meters in one mile.)

$$\left(60 \ \frac{\text{mi}}{\text{h}}\right)\left(\frac{1609 \text{ m}}{1 \text{ mi}}\right)\left(\frac{1 \text{ h}}{60 \text{ min}}\right)\left(\frac{1 \text{ min}}{60 \text{ s}}\right) = 27 \ \frac{\text{m}}{\text{s}}$$

Note that since hours was originally in the *denominator*, we had to make sure to put it in the numerator in a later unit factor to make it go away (since we didn't want any hours in our final answer).

With this simple method, you can convert any quantity from one set of units to another set of units, keeping track of all the conversions as you do so.

5 Converting Text to Equations

What many students find is most difficult with the mathematical problems in this course is figuring out how to take the text of word problems and express them as equations. While the best way to get good at this is practice, there are a few basic things that may help you keep track of what you are doing when trying to express concepts mathematically.

Remember that "is" usually means "equals". Second, remember that "times" usually means multiplication. Take this example sentence: the mass of star A is five times the mass of star B. That sentence is almost a mathematical expression by itself. When you see that, you can write down:

$$M_A = 5\,M_B$$

That's one equation that you know to be true.

Next, when you are asked to compare one thing to another, you are usually being asked for a ratio. How much farther is star A than star B? This means that you are being asked for:

$$\frac{d_A}{d_B}$$

where d_A is the distance to star A and d_B is the distance to star B. Suppose you calculated that ratio to be 3; you would then say "star A is three times farther away than star B".

You can often get a lot of mileage out of breaking down the problem; for each fact the problem gives you, ask yourself if you can write down an equation to express that. Then make sure that you understand how the equations you've written down express either something that is given to you in the problem, or something that you otherwise know (e.g. from the book and from lecture). While sometimes you can get an answer to a problem by playing with the equations, you will learn more (and are more likely to get a right answer) if you understand where you start, and you understand what the answer means when you finish. Indeed, make sure that the answer makes sense; if you calculate the mass of a star, and it comes out at 10kg, does this make sense given what you know about stars?

If you are having difficulty figuring out how to express the text of problems as equations, do not hesitate to ask the TAs or the instructor about this. We know that this can be difficult for those not accustomed to doing scientific problems.

6 Significant Figures

Suppose I tell you that one stick is 1.0 meters long, and that it is 4.7 times longer than another stick. How long is the second stick? Writing the words as equations (see previous section), you might write:

$$l_1 = 4.7\ l_2$$

l_1 is what you know (1.0 meters), and l_2 is what you're looking for, so solve the equation for l_2:

$$l_2 = \frac{l_1}{4.7}$$

plug in the numbers and solve for the answer:

$$l_2 = \frac{1.0\ \mathrm{m}}{4.7} = 0.212765957447\ \mathrm{m}$$

That answer is wrong! Why? Because it is expressed with too many significant figures.

Think about the original problem. I told you a stick was 1.0 meters long. Notice that I didn't say 1.00 meters long; only 1.0 meters long. That means that I was only willing to commit to knowing the length of the stick to within a tenth of a meter. It might really be more like 1.04 meters long, or perhaps 0.98 meters long, but I've rounded to the nearest tenth of a meter. Since I only know the length of the stick to about ten percent, and since I used that number to calculate the length of the second stick, I can't know the legnth of the second stick to the huge precision that I quote above— even though that is the "right" number that my calculator gave me. Given that I only know that the first stick is 1.0 meters long, and it is 4.7 times the length of the second stick, *all that I can say I know* about the length of the second stick is:

$$l_2 = 0.21 \text{ m}$$

This is the reasoning behind significant figures. There are four basic rules of significant figures:

1. When multiplying or dividing numbers, the answer has as many significant digits as that member of the product or quotient that has fewer significant digits. So, if I multiply 3.14159 by 2.0, the answer is 6.3; I round the answer to two significant figures, because 2.0 (the member of the product with fewer significant figures) only has two.

2. When adding or subtracting numbers, the answer is preice to the decimal place of the *least precise* member of the sum. If I add 10.02 meters to 2.3 meters, the answer is 12.3 meters. The second number was only good to the first decimal place, so the sum is only good to the first decimal place. Notice that the *number* of significant figures here is different from either number that went into the sum. When *multiplying*, it is the *number* of significant figures that is important; when *adding*, it is the *decimal place* that is important.

   (Note that if I were to add 10.02 to 2.30 meters, the answer would be 12.32 meters; in this case, both members of the sum are significant to the hundreds place.)

3. A number which is *exact* should not go into considerations of significant figures. For example, suppose you're doing a unit factor conversion, and you multiply by the factor (12 in/1 ft). Your answer need not be limited to two significant figures because of this; there are *exactly* 12 inches in one foot. That's a *definition*; there is no uncertainty associated with it.

4. **Always keep two or three more figures during intermediate calculations than you will report as significant figures in your final answer.** Otherwise, "round-off" errors will accumulate, and you may get the final answer wrong even though your general method and equations were correct.

I am not going to be especially picky about significant figures; don't worry about getting it exactly right. Just be reasonable, and make sure you understand the rationale behind why an answer might have a limited number of significant figures. If you report an answer with one or two too many significant figures, I will *not* take off any points on homework or exam problems. However, if you report something that obviuosly has too many significant digits— i.e., if you report an answer that claims knowledge more precise than what is really availble given the problem— that answer is wrong, and I will take off a small amount of credit for having the wrong number of significant figures.